Closed Geodesics in *S*tair*C*ase *M*etric (*SCM*) manifolds

Geometry and Geometric Analysis (GaGA) Seminar

Tucker Technology Center @ TCU

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via the *SCM Geometry Group*
at Austin College

Abstract

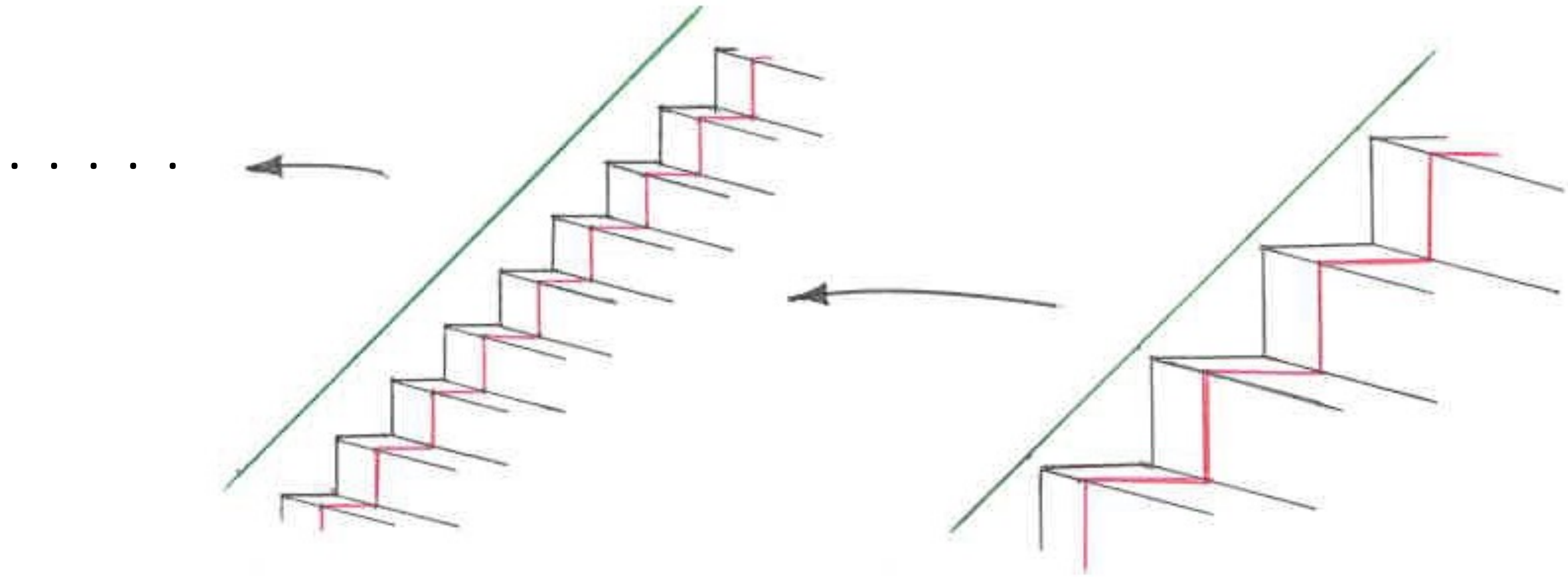
After an introduction to the category, *StairCase Metric (SCM) Geometry*, we focus on defining various SCM manifolds with non-trivial topology. In these we can find the geodesics exactly, then categorize. In our last example, a reasonably natural *space-time* SCM manifold is defined and studied; then various aspects of the *time-like* geodesics are identified.

One motivation for the category of *SCM* manifolds

Comparison. Recall that a smooth sphere can be realized as (countable) limit of a sequence of appropriately defined convex polyhedral surfaces, each *a.e.*-flat and with only a finite number of cone points (vertices).

Think of starting with a large block of wood. With your belt sander, sand away everything *greater* than one unit from the original (and fixed) center of the block. Then rotate the block by any ***rational*** {longitude & latitude}. Repeat sanding, and via the same scheme. Et cetera. So, a *countable* sequence of such sanded solids has as its limit the unit ball; the resulting (smooth) boundary surface is the unit 2-sphere.

In a somewhat similar way, consider an infinite set of steps (see below) abutting a wall; an infinite set of *cone points* comprises the ‘not intrinsically flat’ part of the manifold. Note that the red geodesic is very close to the green geodesic, but significantly longer. Now ‘*cut the step size by half*’ throughout; the same ‘length ratio’ is in play in the resulting manifold. Repeat *ad infinitum*. The *limit space* comprises a simple example of a *staircase metric* (*SCM*) manifold.

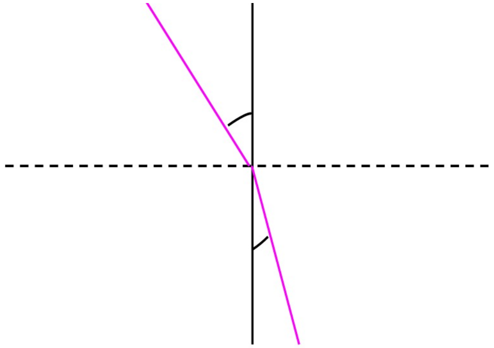


A mild generalization of this example follows. Let C be a complete (and non-self-intersecting) curve in a flat plane. Then *decree* the following metric, where both $\{n_1, n_2\}$ are distinct positive real numbers:

$$ds = \begin{cases} n_1\sqrt{dx^2 + dy^2}, & if\ (x,y)\ is\ \text{"}above\text{"}\ C \\ n_2\sqrt{dx^2 + dy^2}, & if\ (x,y)\ is\ \text{"}below\text{"}\ C \end{cases}$$

It is in this sense that the ***metric*** is '*staircase*' – a step function is involved. Notably, even in this simple case, the defining metric of the manifold is *not* differentiable; indeed, it's not even continuous. Generally, we'll investigate manifolds with more complicated *staircase metrics*. The change in angle (in this case) follows via a familiar equation.

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$



Linguistic note: Occasionally, optical terms such as '*vacuum region*', '*index of refraction*' (*i.o.r.*), '*Snell boundary*', et cetera, may slip into the conversation. However, any relation to optics is neither intended nor needed.

Notation update: Use '*df*' or '*dilation factor*' instead of '*i.o.r.*'. Then '*df-boundary*' is the connecting gizmo, often shown as 'dotted curves' in 2-d manifolds.

Generalizing this example in multiple ways, we (*currently*) define ***staircase metric* (*SCM*) *geometry*** as the study of geometric manifolds with the following characteristics:

1) Said manifold is a real, path connected, topological space.

2) Said manifold consists of sub-region/s, often of local constant curvature *except on* a countable collection of pairwise non-intersecting C^1 (*n-1*)-manifolds, the latter which comprise the boundaries *connecting* the (*n-dimensional*) sub-regions.

3) Further, various quotient schemes may be utilized to path-connect these sub-regions along said boundaries;

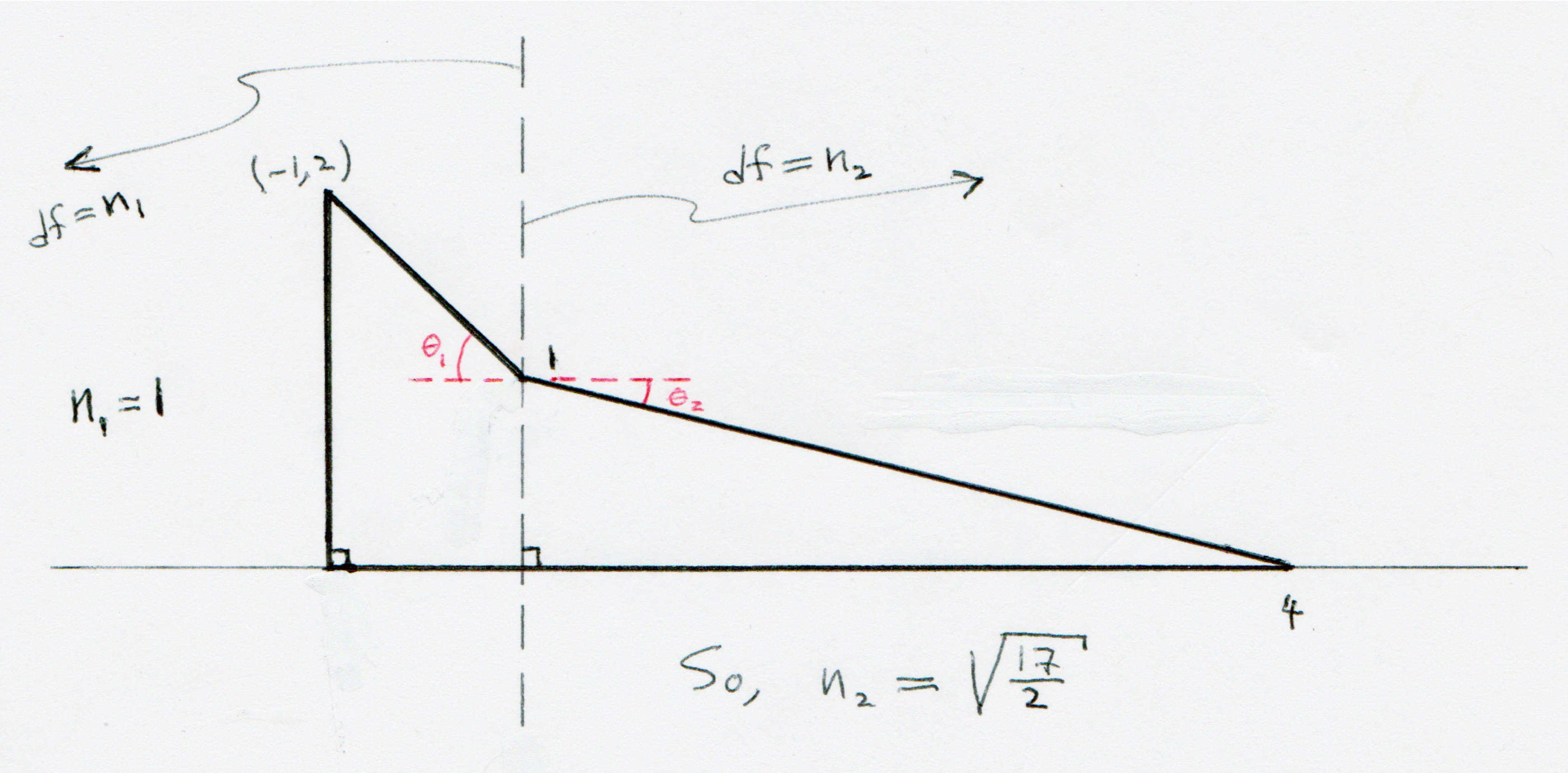
4) Signature (1, n-1) manifolds are included; a typical (though not necessary) restriction is that boundary identifications only occur along space-like (n-1)-*dimensional* sub-region boundaries.

The metrics on each sub-region ***may*** be identical *except* for a scaling; in such cases, constant scale factors - *dilation factors/ df'*s - are used to distinguish the subregions. An appropriate *angle change law* governs the *geodesic* paths that cross the various (*n-1*)-dimensional boundaries between these regions.

<u>*Key*</u>: The freedom to set, independently and in varying orders {the number (often infinite) and shapes of the various map regions; the *df*'s of the map regions; the quotient scheme applied to the boundaries of the map regions; the curvatures of the underlying map regions; the signature of the space} provides a distinct and reasonably versatile alternative to the scheme, "*Consider the following smooth metric*."

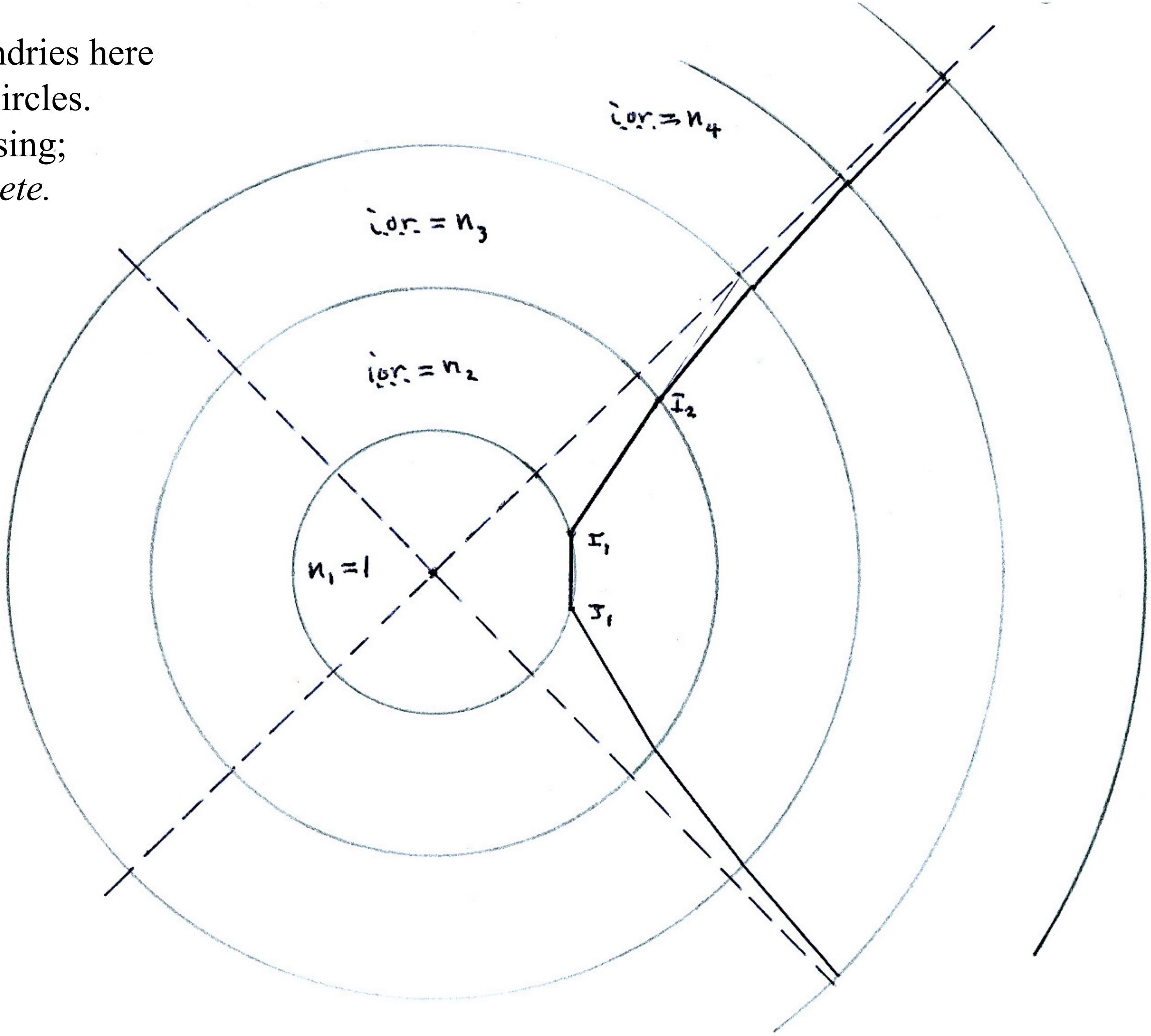
Example **1**: Set the left side and the base of the 3-gon as shown. The second ***dilation factor*** (or *conformal factor*) is ***chosen*** so that the third side of the triangle extends to the predetermined vertex. Invoke the a.c.l., $n_1\, sin(\theta_1) = n_2\, sin(\theta_2)$; then, with the angles being as shown, solve for n_2 . Note: Below is the *parameter space* view.

Nb: the *y*-axis is the *df-boundary for this manifold.*



(N.b.: *example* **2** follows.)

<u>Example</u> **2**. The *df*-boundries here are concentric (planar) circles. The *df* values are increasing; the *geodesic is bi-complete.*



Three related, but substantially distinct SCM tori

It is natural and of considerable interest to consider SCM manifolds with more complex topology.

In the two dimensional category, we proceed to define three distinct, *almost everywhere* constant curvature, genus-one SCM tori.

The constructions are reasonably natural; the resulting manifolds feature interesting properties, which we compare.

Properties of the aforementioned genus-1 SCM tori

Rk #1: Each of the three tori feature exactly one *df*-boundary; said boundaries are *topological* circles. Then *a.c.l.*'s are derived for each.

Rk #2: Away from the *df*-boundary, each has ***constant*** G-curvature.

Rk #3: All are *a.e.* pairwise distinct w.r.t. *Gaussian curvature* (on the 'fat' parts); that is: {zero; positive; negative}.

Rk #4: All 3 constructions are, we claim, reasonably *natural*.

Rk #5: Of note, there are no ad hoc dilation factors per se; rather, we utilize *natural* df's that stem purely from the boundary quotient scheme.

Sub-ex's 3.1: Tori utilizing annular subsets of the *Euclidean plane*.

A genus-1 torus that is *almost everywhere* (*a.e.*) **flat** can be constructed easily via the SCM scheme, as the following diagram illustrates. The two boundaries are ***identified*** by decreeing that 'equal times are the same point' – e.g., 3:19:57 o'clock is the same point on both the inner and outer circles. So, an '*a.e.*-flat' torus results. (The radii: r; R.)

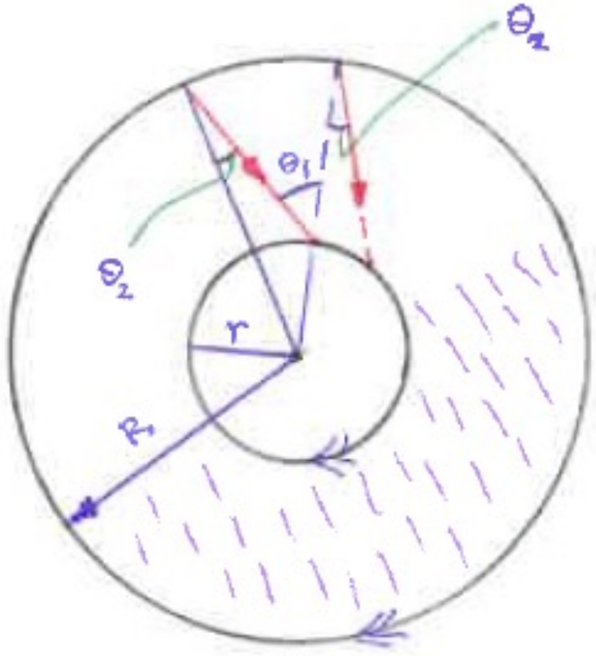
Prop 3.1.1: The effective ***angle change*** condition here is:

$$r \sin(\Theta_1) = R \sin(\Theta_2)$$

Prop 3.1.2: $\exists \geq 3$ types of geodesics in this genus 1 *SCM* manifold:

1. bi-complete geodesics.
   rk. closed geodesics v. not closed
2. non-extendible geodesics.
3. extendable, but ***not*** uniquely.

Observation: $\mathcal{X}$(this manifold) = 0
Q: What is the boundary curvature?



Rk: the boundary quotient is "radial".

Obs: $1 \sin(\theta_1) = \frac{R}{r} \sin(\theta_2)$

So: $\boxed{r \sin(\theta_1) = R \sin(\theta_2)}$

Sub-ex's 3.2: Tori via 'temperature zone' subsets of a *sphere*.

A genus-1 torus that features *a.e.*- ***positive***, constant Gaussian curvature can be constructed easily via the SCM scheme, as the next diagram will illustrate.

Regarding the boundary quotient: The two boundaries, which are two *latitude lines* on a sphere, are ***identified*** by decreeing that two points on these two latitude lines, that also are of *equal longitude*, ***are the same***.

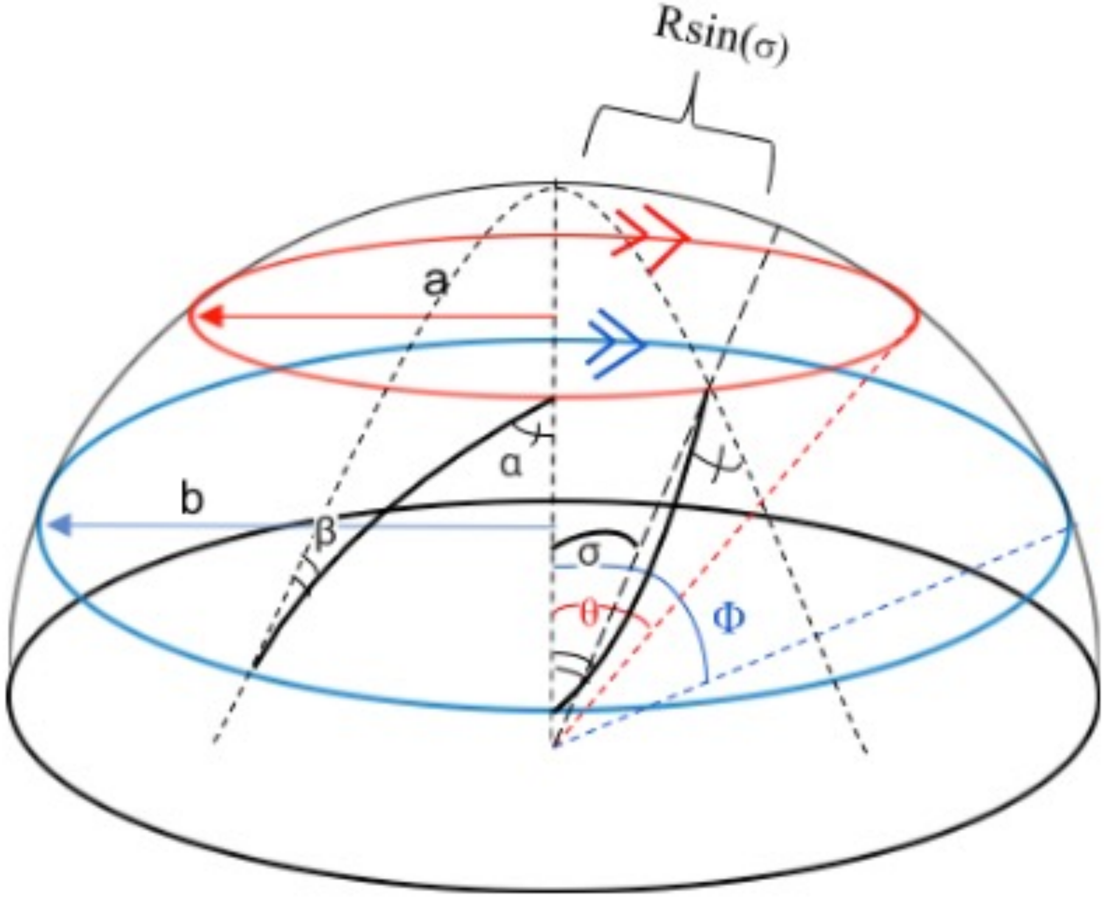
Geodesics on a.e. positive curvature torus

- While there exists only one df on this manifold, due to the boundary quotient scheme the classical ACL is in effect.
- **<u>Proposition 2a</u>**: The *a.c.l.* for this torus is:

$$a \sin(\alpha) = b \sin(\beta)$$

- Any geodesic on the region can be defined by its shortest distance from the north pole, an arc of length $R\sigma$, perpendicular to the geodesic.
- **<u>Proposition 2b:</u>** All geodesics can be classified into one of three types. (Proof ***omitted***.)
- <u>Rk</u>: Don't have to utilize the entire zone. Instead, z.b., use only part of the time zone, with the longitude quotient being smooth.

- **Type 1: $\sigma < \theta$, uniquely extendable**



Sub-ex's 3.3, and <u>via</u> the PUHP model of the *hyperbolic plane*.

Consider a (2-d) ***torus***, but constructed via the usual *Poincare' upper-half plane*. In this model,

$$ds = \frac{1}{y}\sqrt{dx^2 + dy^2} \quad .$$

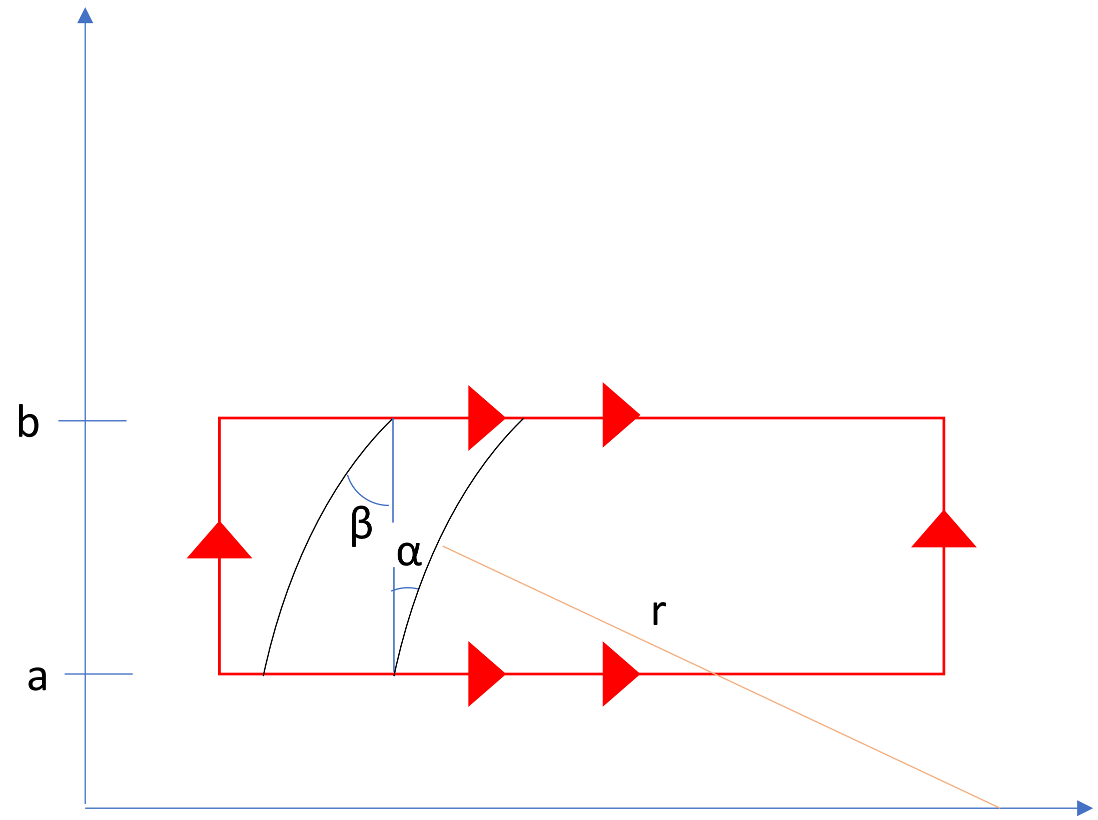
Geodesics in the PUHP appear as *vertical lines* <u>or</u> *arcs of circles centered on the* x-*axis*.

But consider a (*parameter space*) rectangle of some arbitrary width, ***d***; further, the lower side is at height **a**, and the upper side is at height **b** above the axis (b > a). <u>Identifying the left and right sides of this rectangle is natural</u>; so, *no angle change* results due to that quotient. But now identify the upper/lower sides via:

for all *x*, (*x*, *a*) ~ (*x* , *b*) .

The resulting *angle change law* for a geodesic crossing the horizontal boundary follows:

<u>*PROP 3.3.1*</u>: a.c.l. is: (1/*b*) *sin*(β) = (1/*a*) *sin*(α)



(Sub-ex's 3.3, continued)

Prop 3.3.2: There are 3 types of geodesics in this genus 1 *SCM* manifold with a.e. ***constant negative*** G-curvature.

- 1.A: bi-complete geodesics.
- 1.B: closed geodesics.
- 2: non-extendable geodesics.
- 3: bifurcating geodesics.

Rk: It's likely there are yet finer classifications of these geodesics.

The above examples are in the positive definite category; the underlying parameter space *sub*-regions were among subsets of {flat planes with the usual Euclidean metric}, {2-spheres}, and {hyperbolic plane parameter spaces}. In these manifolds, the geodesics are *everywhere locally length minimizing*; also, said parameter spaces are *conformal*.

More recent work involves constructions that are in the <u>space-time</u> category, hence with signature (1,q). [(*time, space*).] Notably, in these cases, the *time-like* geodesics are *everywhere locally* <u>***time*** *maximizing*</u>. In what follows, we utilize *piece-wise* Lorentzian (*flat* 2-d space-time) regions in the construction of the manifold.

Before introducing a specific pseudo-*H*^(1,1) manifold, a quick reminder of a couple of key dynamics in special relativity.

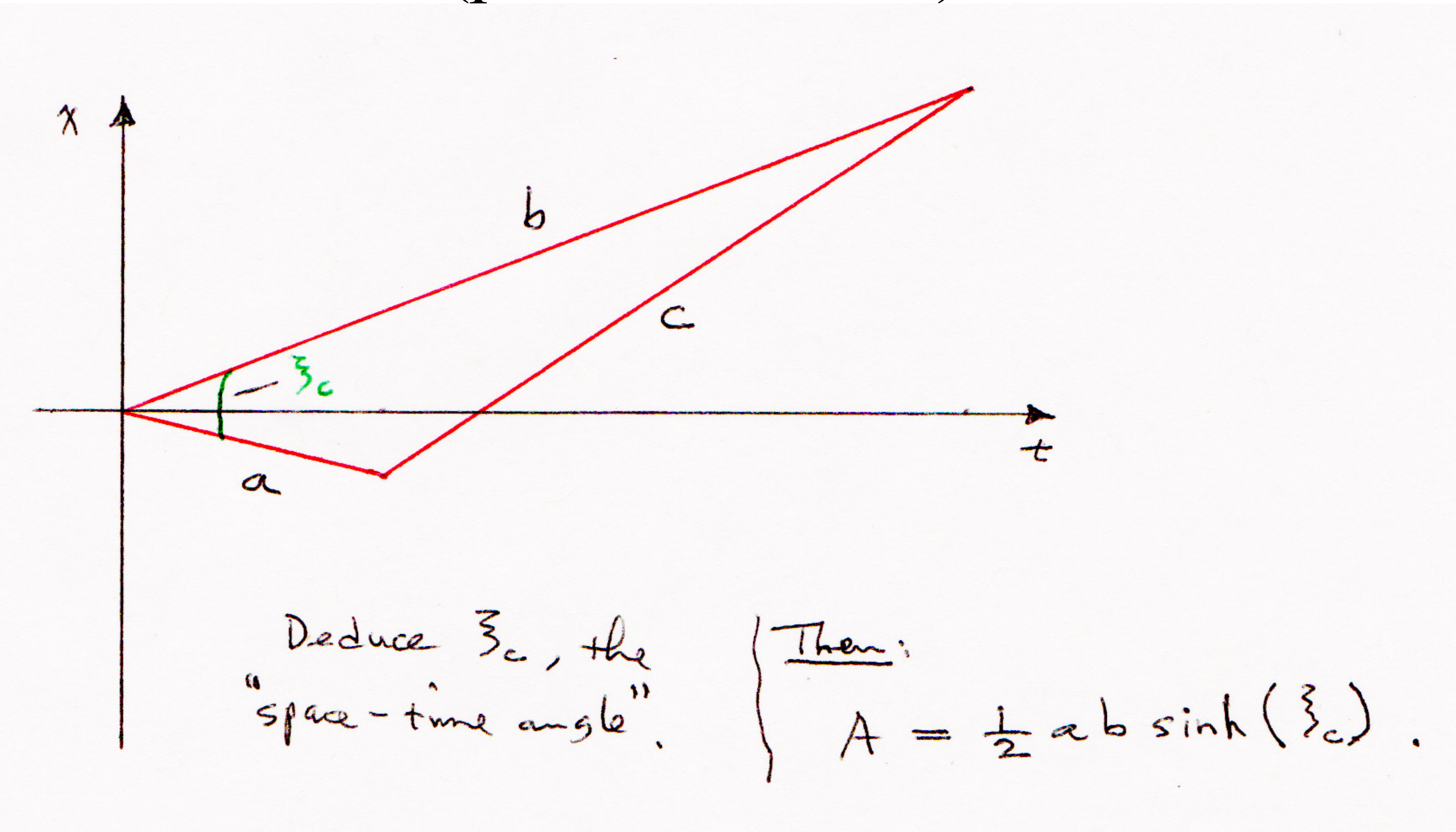
In the space-time category, we restrict attention to time-like vectors, as suggested in the picture. We further consider *stangles* (the space-time analogue of *angles*) where one deduces, for example, the space-time area formula analogous to the classical (positive definite) formula.

Note: a *stangle*, ξ , can take any real value; further, the relation of any *stangle*, ξ , to the usual notion of *angle*, θ , is given by the following:

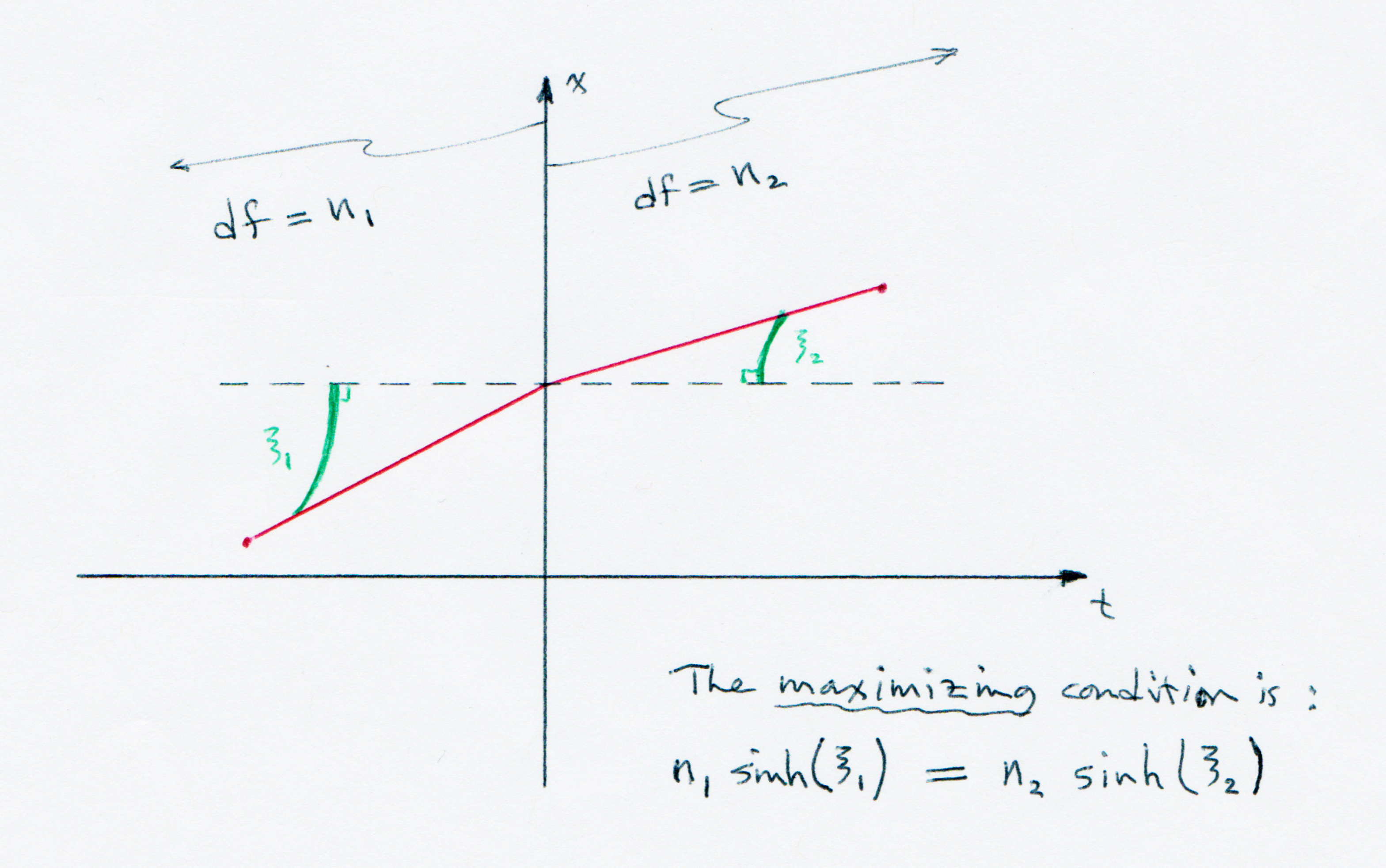
$$\text{angle:} \quad \frac{-\pi}{4} < \theta < \frac{\pi}{4}$$
$$\text{stangle:} \quad -\infty < \xi < \infty$$

Key id: $tan(\theta) = \frac{x}{t} = tanh(\xi)$

(standard position assumed here.)



Utilizing this notion of "*stangle*", one can deduce the following *stangle change law* (analogous to the classical version) for *maximizing* time-like geodesics.



Prop: The a.c.l. for time-like geodesics in this manifold is:

$$n_1 \sinh(\xi_1) = n_2 \sinh(\xi_2)$$

Note that the vertical axis is the *df*-boundary here. Also, the dotted line is s.t.-normal to the df-boundary.

Note on orthogonality in Minkowski space

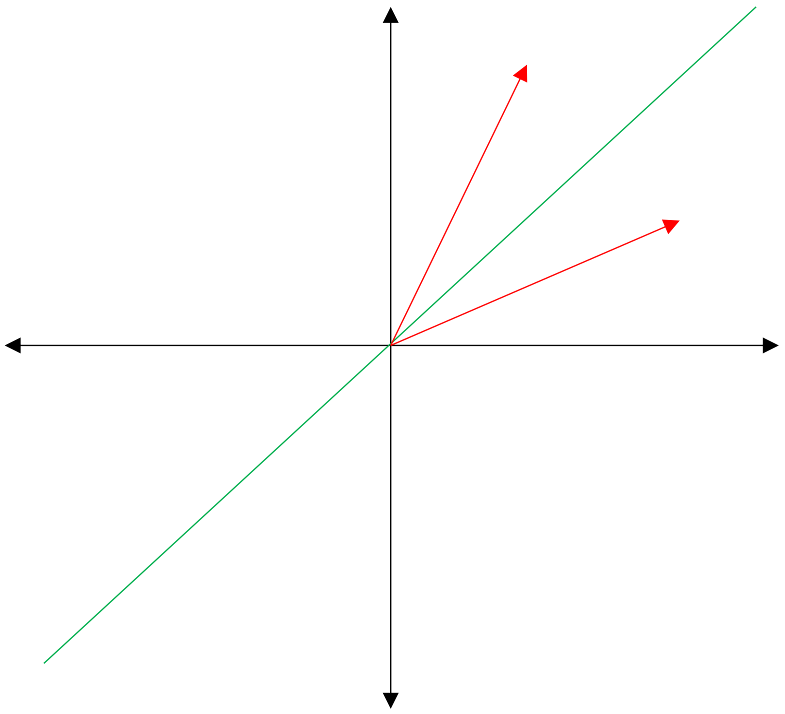
Orthogonality, by definition, requires that the inner product of two orthogonal vectors be zero. In positive definite space, this can be written as:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

However, in Minkowski space, the inner product is written:

$$\vec{a} \cdot \vec{b} = a_1 b_1 - a_2 b_2$$

Note that two orthogonal vectors in this space are reflections of each other about the light-like vector in whichever given quadrant they exist.

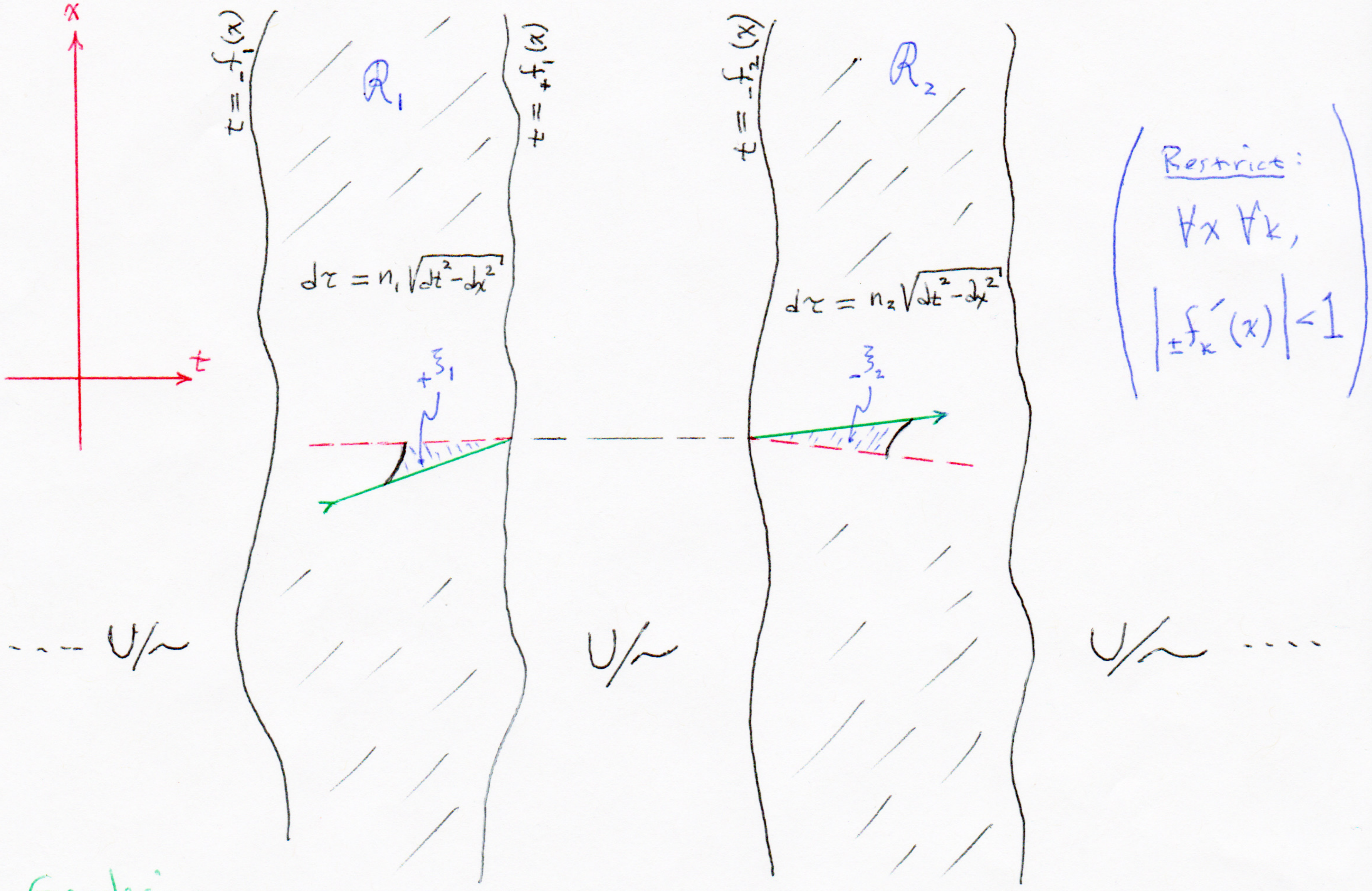


Observe that one can generalize the boundaries in these (1,1) cases in much the same way as in the positive definite case. One way to do this is via a countable set of *space-like* boundary quotients.

Recall that the we take, for convenience, the time direction in the (1,1) manifolds to be horizontal; the sub-region boundaries must all be completely space-like.

See the following diagram. Note that the a.c.l. for this situation is very similar to that in the positive definite case.

One possible general boundary quotient scheme for a.e. flat space-time (1,1) SCM manifolds



Geodesic Maximizing Condition:

$$n_1 \sinh({}_{+}\xi_1)\sqrt{1 - ({}_{+}f_1'(x))^2} = n_2 \sinh({}_{-}\xi_2)\sqrt{1 - ({}_{-}f_2'(x))^2}$$

Various geodesics in the *pseudo unit* $H^{1,1}$ *staircase metric* (SCM) *manifold*

Said manifold is topologically a cylinder. The time direction involves a natural quotient of the two *space-like* boundaries; this quotient yields the staircase metric dynamic for this spacetime. The spatial direction is infinite; any *time-like* path will cross the space-like *df*-boundary infinitely many times.

The manifold:

The *left* boundary: the (vertical) *x-axis*.

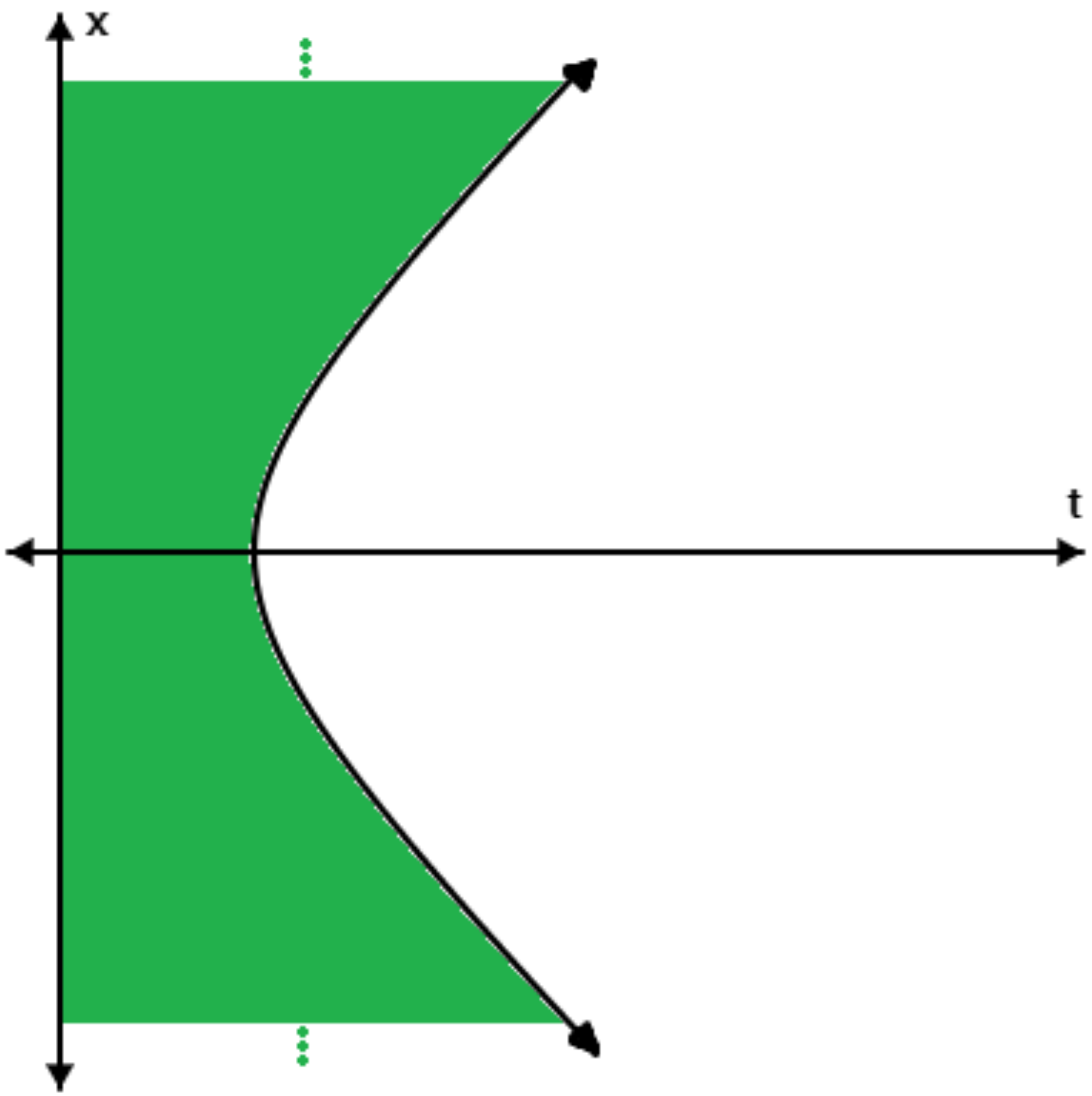
The *right* boundary: $t = \sqrt{1 + x^2}$

The space-time *SCM* manifold of interest is obtained by identifying the two vertical (space-like) boundaries via the quotient:

for all *x*, $(0, x) \sim (\sqrt{1 + x^2}\,, x)$.

So, there is exactly *one df*-boundary in this SCM-manifold. Also note: the time-like geodesics will always impact the *df*-boundary infinitely many times.

"Idealized infinite trans-Chilean railroad."



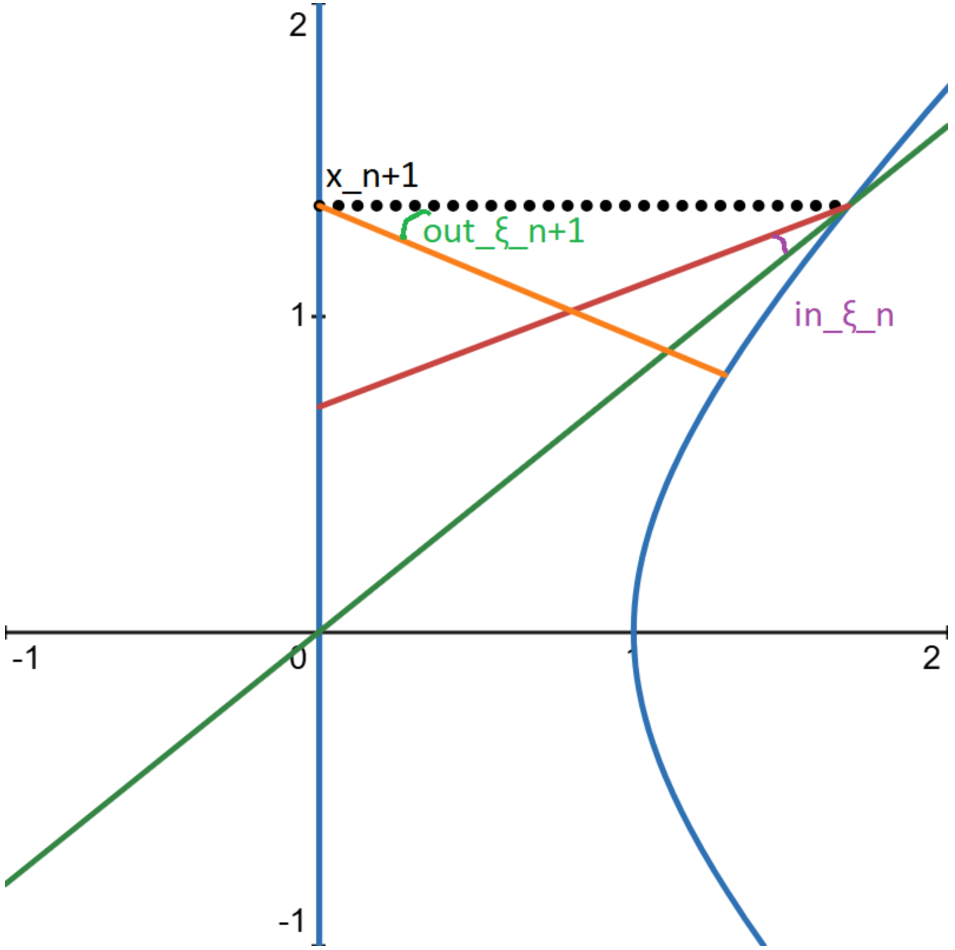
Terminology

- **ST-Angle:** Short for Space-Time Angle. ξ is the symbol for st-angles
- **Clean:** The geodesic contains some x_n such that $x_n = 0$
- **Dirty:** The geodesic does not contain some x_n such that $x_n = 0$
- **Bounded:** The geodesic does not go off to infinity in the spatial direction.
- **Closed:** The geodesic loops
- **nth Time Circuit:** The segment of the geodesic that goes from $(0, x_{n-1})$ to (t_n, x_n)
- **Period:** The number of time circuits it takes for the geodesic to loop
- **Even Geodesic:** The geodesic has an even period
- **Odd Geodesic:** The geodesic has an odd period
- **Trivial Geodesic:** The geodesic that contains $\xi_n = 0$ and $x_n = 0$ for all *n*.

Geodesics in the manifold: Prop. 1.0, the ‘*stangle*’ change law for this manifold

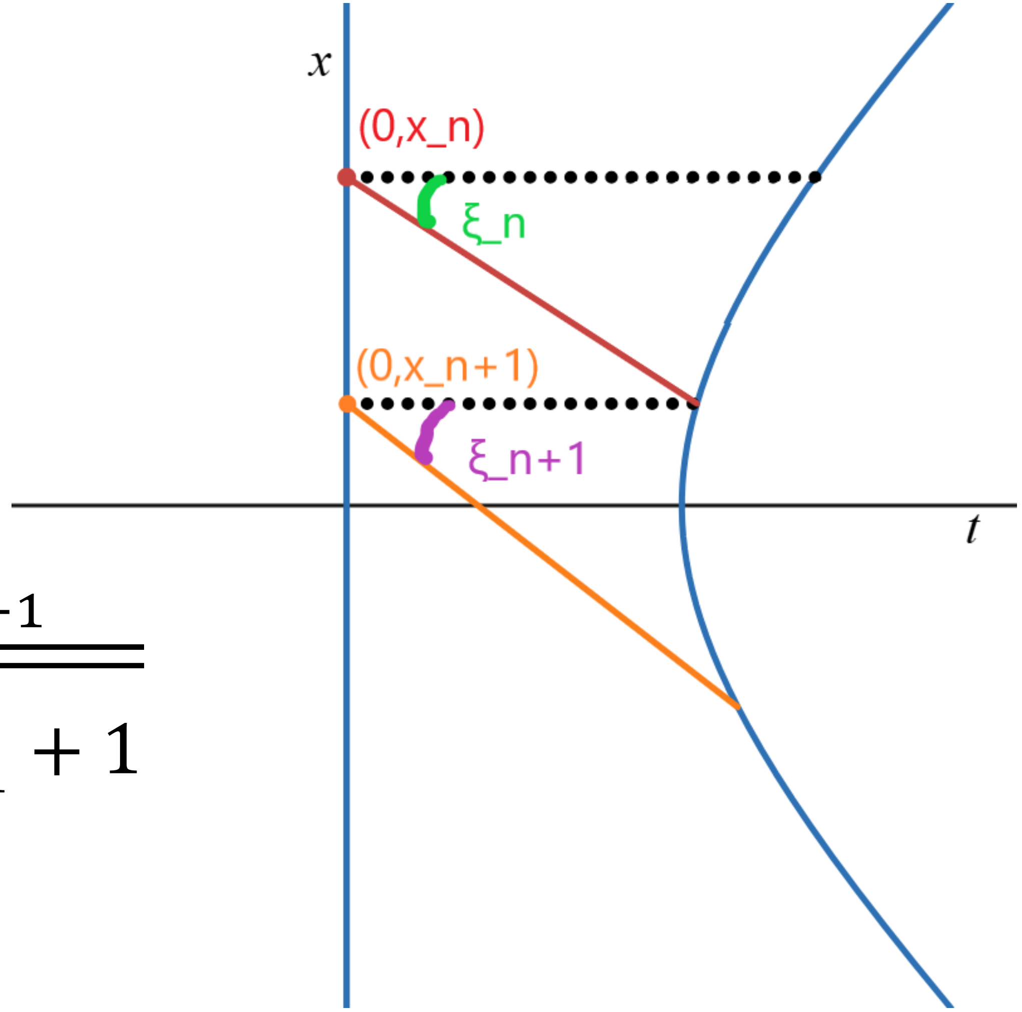
The green line is orthogonal to the point in which the “in” time circuit (red) intersects the righthand boundary.

$$sinh({}_{out}\xi_{n+1}) = \frac{1}{\sqrt{x_{n+1}^2 + 1}} sinh({}_{in}\xi_n)$$



Geodesics in the Manifold: Proposition 2

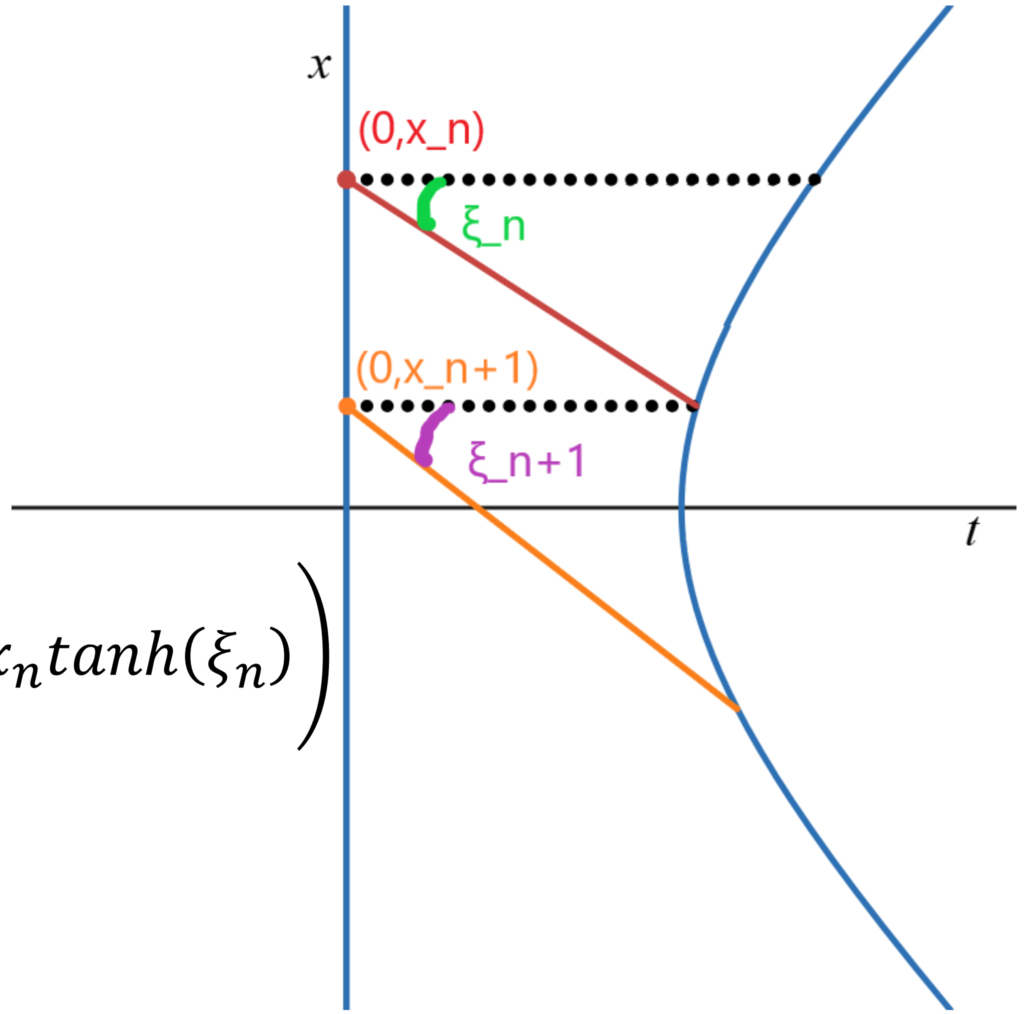
Let ξ_n and ξ_{n+1} be the st-angle of the nth and n+1th time circuit respectively and let x_{n+1} be the x value where the st-angle change takes place



$$sinh(\xi_{n+1}) = sinh(\xi_n) - cosh(\xi_n)\frac{x_{n+1}}{\sqrt{x_{n+1}^2 + 1}}$$

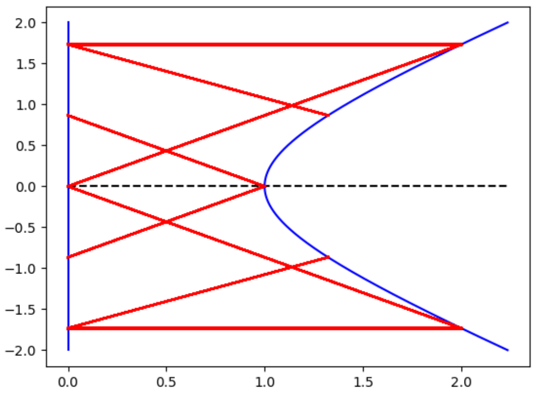
Geodesics in the Manifold: Proposition 3

Let ξ_n and x_n be the associated st-angle and x value with the nth time circuit and let x_{n+1} be the x value of the n+1th time circuit.

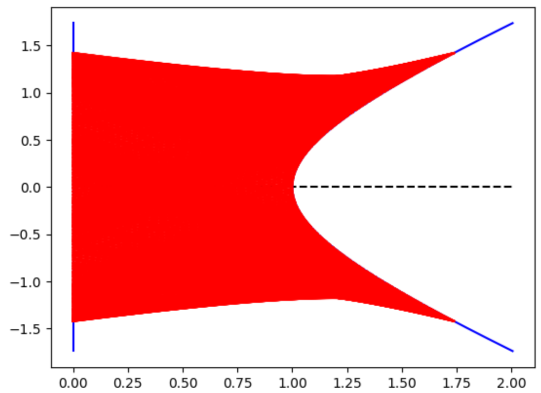


$$x_{n+1} = x_n + \frac{1}{2} sinh(2\xi_n)\left(\sqrt{sech^2(\xi_n) + x_n^2} + x_n tanh(\xi_n)\right)$$

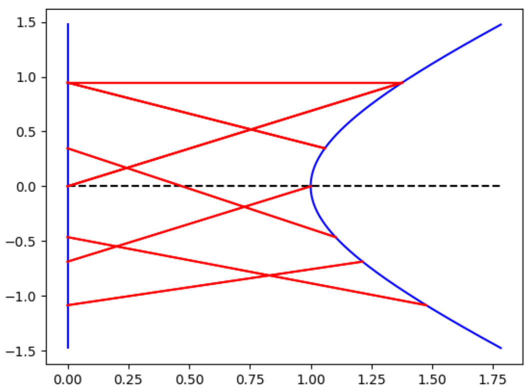
Geodesics and Theorized Geodesics: Clean



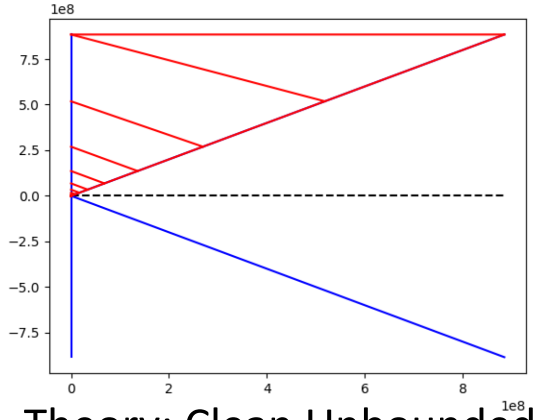
Clean Bounded Closed Even



Theory: Clean Bounded Non-Closed

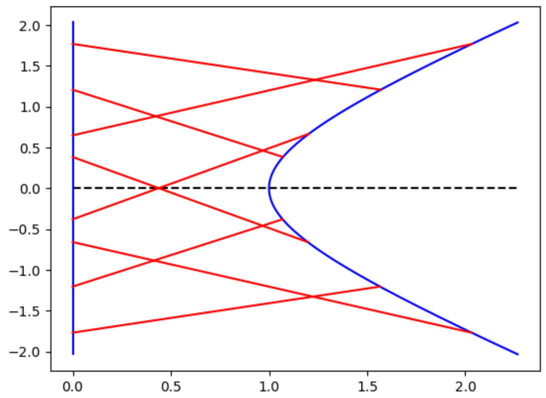


Clean Bounded Closed Odd

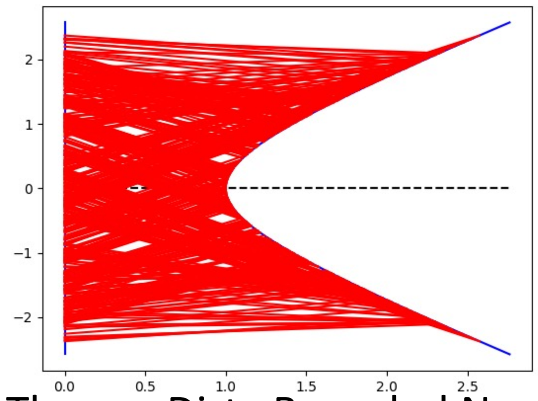


Theory: Clean Unbounded

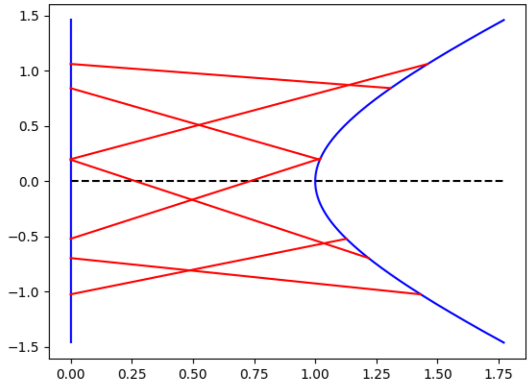
Geodesics and Theorized Geodesics: Dirty



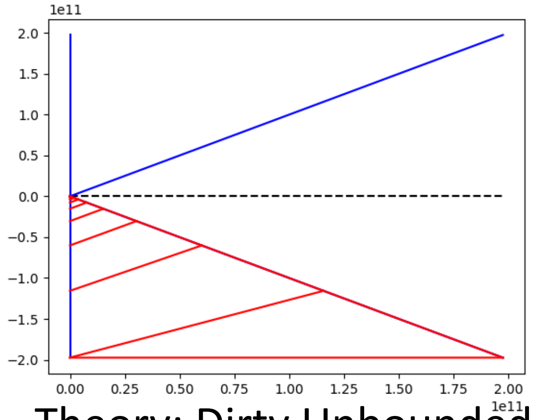
Theory: Dirty Bounded Closed Even



Theory: Dirty Bounded Non-Closed



Theory: Dirty Bounded Closed Odd



Theory: Dirty Unbounded

Conjectures

- **Conjecture 1:** A clean closed geodesic of period 2a can be generated from $x_0 = 0$ and $x_a = 0$ where ξ_0 and ξ_a do not share the same sign.
- **Conjecture 2:** A clean geodesic is closed if and only if there exists 2 x values that equals 0 in the geodesic.
- **Conjectures 3 and 4:** Clean geodesics with periods of 4 and 6 time circuits do not exist.
- **Conjectures 5, 6, and 7:** Clean geodesics with periods of 2, 3, and 5 time circuits do not exist.
- **Conjecture 8:** The smallest number of time circuits for a non-trivial clean closed geodesic is 7.

Lemmas

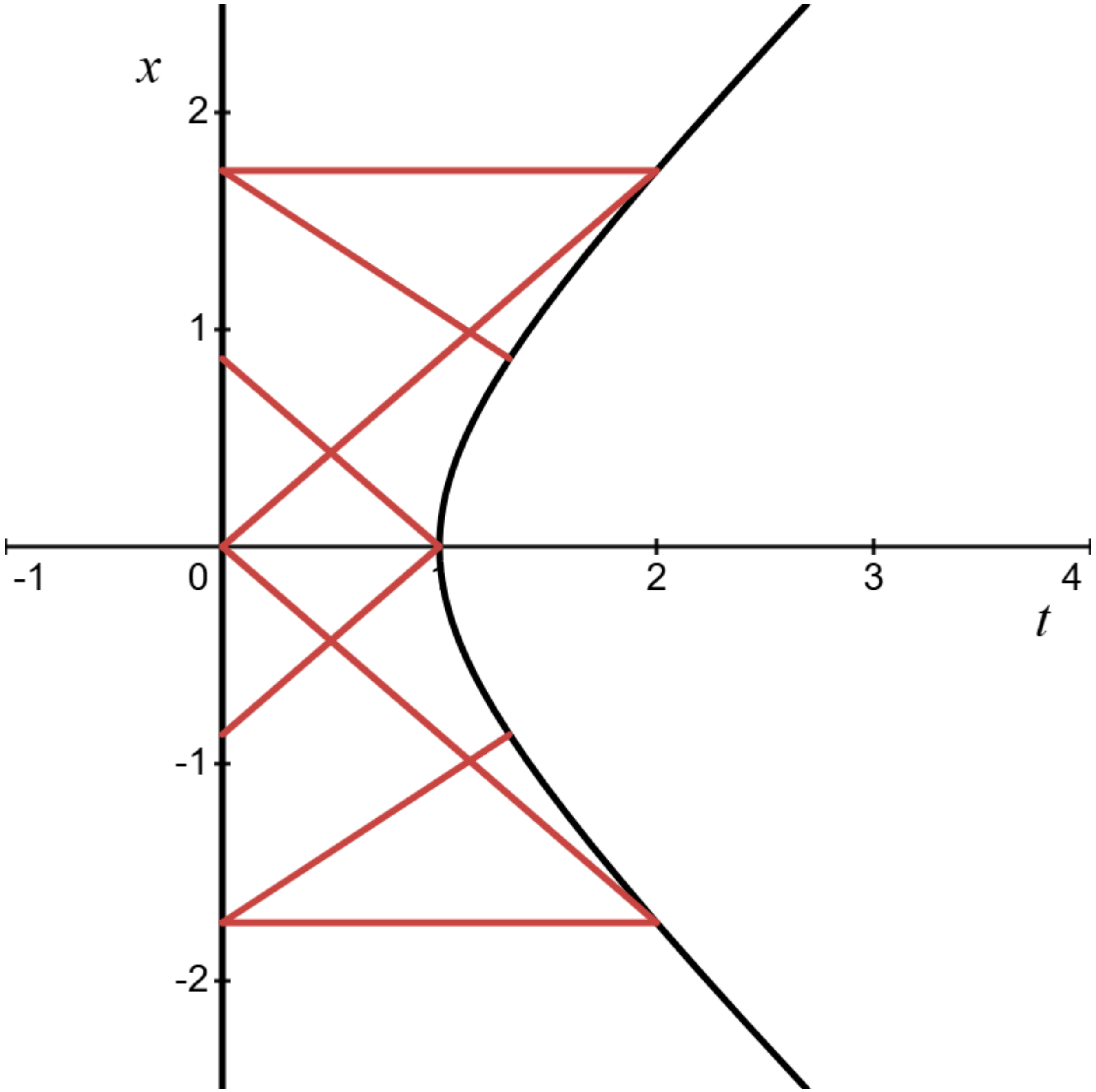
- **Lemma 1:** A geodesic with $x_0 = 0$ and ξ_0 has a geodesic that is mirrored over the t-axis that can be generated by taking $x_0 = 0$ with $-\xi_0$.
- **Lemma 2:** If x_n and x_{n+1} have the opposite sign, then ξ_n and ξ_{n+1} have the same sign.

Theorem 1

The following are the starting *x* and ξ values for the 8 – period closed clean geodesic.

$$x_0 = 0\,; \quad \xi_0 = arcsinh(\sqrt{3})$$

This was proven algebraically with Lemma 1.



10 – Period Closed Clean Geodesic

The solution to this equation will give us x_1 for the 10 – period clean closed geodesic.

$$-\left(x_1-\frac{x_1}{\sqrt{1+x_1^2}}\left(\sqrt{x_1^2\left(1+\frac{x_1^2}{1+x_1^2}\right)+1}-\frac{x_1^2}{\sqrt{1+x_1^2}}\right)\right)+\frac{x_1}{\sqrt{1+x_1^2}}=-\frac{\left(x_1-\frac{x_1}{\sqrt{1+x_1^2}}\left(\sqrt{x_1^2\left(1+\frac{x_1^2}{1+x_1^2}\right)+1}-\frac{x_1^2}{\sqrt{1+x_1^2}}\right)\right)\sqrt{1+\frac{x_1^2}{1+x_1^2}}}{\sqrt{1+\left(x_1-\frac{x_1}{\sqrt{1+x_1^2}}\left(\sqrt{x_1^2\left(1+\frac{x_1^2}{1+x_1^2}\right)+1}-\frac{x_1^2}{\sqrt{1+x_1^2}}\right)\right)^2}}$$

$$x_1=0\,;\quad x_1\approx\pm 3.9495791311069649279\ldots$$

Approximate n Period Closed Clean Geodesics

- 7 Period: $\xi_0 \approx 0.84268$
- 9 Period: $\xi_0 \approx 1.639067$
- 10 Period: $\xi_0 \approx 2.0824$
- 11 Period: $\xi_0 \approx 2.3245$
- 12 Period: $\xi_0 \approx 2.795734$
- 13 Period: $\xi_0 \approx 2.98723$
- 14 Period: $\xi_0 \approx 3.4948316$

- 7 Period: $x_0 \approx 0.946$
- 9 Period: $x_0 \approx 2.478$
- 10 Period: $x_0 \approx 3.94957$
- 11 Period: $x_0 \approx 5.0592$
- 12 Period: $x_0 \approx 8.1467$
- 13 Period: $x_0 \approx 9.891$
- 14 Period: $x_0 \approx 16.457$

What about higher dimensional versions?
***Answer*: Stay tuned!!!**

***Questions*?**

***Thanks*!**

(See acknowledgements following. **)**

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